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Integral Equations and Operator Theory

On Pólya's Inequality for Torsional Rigidity and First Dirichlet Eigenvalue

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Abstract. Let Ω be an open set in Euclidean space with finite Lebesgue measure $|\Omega|$. We obtain some properties of the set function $F : \Omega \mapsto \mathbb{R}^+$ defined by

$$F(\Omega) = \frac{T(\Omega)\lambda_1(\Omega)}{|\Omega|},$$

where $T(\Omega)$ and $\lambda_1(\Omega)$ are the torsional rigidity and the first eigenvalue of the Dirichlet Laplacian respectively. We improve the classical Pólya bound $F(\Omega) \le 1$, and show that

$$F(\Omega) \le 1 - \nu_m T(\Omega)|\Omega|^{-1-\frac{2}{m}},$$

where ν_m depends only on m. For any $m = 2, 3, \ldots$ and $\epsilon \in (0, 1)$ we construct an open set $\Omega_\epsilon \subset \mathbb{R}^m$ such that $F(\Omega_\epsilon) \ge 1 - \epsilon$.

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1. Introduction

Let Ω be an open set in $\mathbb{R}^m$ with finite Lebesgue measure $|\Omega|$, and let $v_\Omega : \Omega \mapsto \mathbb{R}^+$ denote the corresponding torsion function, i.e. the unique solution of

$$-\Delta v = 1,\ v \in H_0^1(\Omega). \tag{1.1}$$

The torsional rigidity of Ω is defined by $T(\Omega) = \int_\Omega v_\Omega$. As $v_\Omega \ge 0$, the torsional rigidity is the $\mathcal{L}^1(\Omega)$ norm of v_Ω. The following variational characterisation is well known

$$T(\Omega) = \sup_{w \in H_0^1(\Omega)\setminus\{0\}} \frac{\left(\displaystyle\int_\Omega w\right)^2}{\displaystyle\int_\Omega |Dw|^2}.$$

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The torsional rigidity plays a key role in different parts of analysis. For example the torsional rigidity of a cross section of a beam appears in the computation of the angular change when a beam of a given length and a given modulus of rigidity is exposed to a twisting moment [2,14]. It also arises in the calculation of the heat content of sets with time-dependent boundary conditions [3], in the definition of gamma convergence [6], and in the study of minimal submanifolds [11]. Moreover, $T(\Omega)/|\Omega|$ equals the expected lifetime of Brownian motion in Ω when averaged with respect to the uniform distribution over all starting points $x \in \Omega$.

Since Ω has finite Lebesgue measure the Dirichlet Laplacian acting in $\mathcal{L}^2(\Omega)$ has compact resolvent. We denote the eigenvalues and a corresponding orthonormal basis by $\lambda_1(\Omega) \le \lambda_2(\Omega) \le \ldots$, and $\{\varphi_1, \varphi_2, \ldots\}$ respectively. Recall the following variational characterisation.

$$\lambda_1(\Omega) = \inf_{z \in H_0^1(\Omega)\backslash\{0\}} \frac{\displaystyle\int_\Omega |Dz|^2}{\displaystyle\int_\Omega z^2}.$$

A classical inequality of Pólya [14], asserts that the function F defined by

$$F(\Omega) = \frac{T(\Omega)\lambda_1(\Omega)}{|\Omega|} \tag{1.2}$$

satisfies

$$F(\Omega) \le 1. \tag{1.3}$$

We note that F is scale independent i.e. for any homothety $\alpha\Omega, \alpha > 0$, of Ω we have that $F(\alpha\Omega) = F(\Omega)$.

The main results of this paper are the following.

Theorem 1.1. *For any open set Ω with finite Lebesgue measure*

$$F(\Omega) \le 1 - \frac{2m\omega_m^{2/m}}{m+2} \frac{T(\Omega)}{|\Omega|^{1+\frac{2}{m}}},$$

where ω_m is the measure of the ball with radius 1 in $\mathbb{R}^m$.

Theorem 1.2. *Let $m = 2, 3, \ldots$. For every $\epsilon > 0$ there exists an open connected set $\Omega_\epsilon \subset \mathbb{R}^m$ depending on ϵ such that*

$$F(\Omega_\epsilon) \ge 1 - \epsilon. \tag{1.4}$$

Corollary 1.3. *The variational problem*

$$\sup\{F(\Omega) : \Omega \text{ open in } \mathbb{R}^m,\ |\Omega| = 1\}$$

does not have a maximiser.

The proof of this corollary is immediate. Indeed, by Theorem 1.2, and Pólya's inequality the above supremum equals 1. Suppose there exists an open set Ω with $F(\Omega) = 1$ and $|\Omega| = 1$. Then Ω has strictly positive torsional rigidity, and $F(\Omega) < 1$ by Theorem 1.1 which is a contradiction.

It was shown in [4, Remark 2.4] that

$$\inf\{F(\Omega) : \Omega \text{ open in } \mathbb{R}^m,\ |\Omega| = 1\} = 0. \tag{1.5}$$

However, for the restriction of $F(\cdot)$ to the class of convex sets in $\mathbb{R}^m$, we have the following.

Theorem 1.4. (i)

$$\inf\{F(\Omega) : \Omega \,\text{open, convex in}\, \mathbb{R}^m,\ |\Omega| = 1\} \geq \frac{\pi^2}{4m^{m+2}(m+2)}. \tag{1.6}$$

(ii)

$$\inf\{F(\Omega) : \Omega \text{ open, convex in}\, \mathbb{R}^2,\ |\Omega| = 1\} \geq \frac{\pi^2}{48}. \tag{1.7}$$

Theorem 1.2 disproves the conjecture in [4] that $F(\Omega) \leq \frac{\pi^2}{12}$. Theorem 1.5 below goes some way towards proving the $\pi^2/12$ bound for open bounded, planar, convex sets. In order to state our main result for convex sets, we introduce the following notation. For a convex set with finite measure, we denote by w the minimum width of Ω (or simply the width of Ω), which is obtained by minimising among all pairs of parallel supporting hyperplanes of Ω the distance between such hyperplanes. The projection of Ω onto one of the minimising hyperplanes is denoted by E. The first eigenvalue of the $(m-1)$-dimensional Dirichlet Laplacian acting in $\mathcal{L}^2(E)$ is denoted by Λ.

Theorem 1.5. (i) *If Ω is an open, bounded, convex set in $\mathbb{R}^m$ with w and Λ as above, then*

$$F(\Omega) \leq \frac{\pi^2}{12}\left(1 + \frac{3c}{2} + \frac{3c^2}{4} + \frac{c^3}{8}\right), \tag{1.8}$$

where

$$c = \left(\frac{32w^2\Lambda}{\pi^2}\right)^{1/3}. \tag{1.9}$$

(ii) *If Ω is an open, bounded, convex set in $\mathbb{R}^2$, then*

$$F(\Omega) \leq 1 - \frac{1}{11560}. \tag{1.10}$$

Corollary 1.6. *If (Ω_n) is a sequence of bounded convex sets with corresponding sequences (w_n) and (Λ_n) such that $\lim_{n\to\infty} w_n^2\Lambda_n = 0$, then*

$$\limsup_{n\to\infty} F(\Omega_n) \leq \frac{\pi^2}{12}.$$

The main idea in the proof of Theorem 1.2 is that if Ω is an open, bounded and connected set, then we can find $x_0 \in \Omega$ and $\delta > 0$ such that punching a hole in Ω centered at x_0 with radius δ increases F. In the proof of Theorem 1.2, we take an m-dimensional cube with side-length L and punch N^m holes with the same radius δ in a periodic arrangement. We show that we can find L, N, δ depending on ϵ (and m) such that the corresponding value of F for the punched cube exceeds $1-\epsilon$. As mentioned above, F is invariant under homotheties, and so we could have chosen $L = 1$. However, it is convenient to keep L undetermined so that we have a homothety or scaling check in the various bounds.

To see that punching a hole increases F, we take Ω open, bounded, connected, and with smooth boundary. Let $\varphi_1 \in H_0^1(\Omega)$ be a Dirichlet eigenfunction corresponding to $\lambda_1(\Omega)$, and let v_Ω be the solution of (1.1). We observe that

$$\lambda_1(\Omega) < \frac{\|Dv_\Omega\|^2_{\mathcal{L}^2(\Omega)}}{\|v_\Omega\|^2_{\mathcal{L}^2(\Omega)}},$$

implies

$$\begin{aligned}\int_\Omega \left(T(\Omega)\varphi_1^2 - \lambda_1(\Omega)v_\Omega^2\right) &= \|v_\Omega\|^2_{\mathcal{L}^2(\Omega)}\left(\frac{1}{\|v_\Omega\|^2_{\mathcal{L}^2(\Omega)}}\int_\Omega v_\Omega - \lambda_1(\Omega)\right)\\ &= \|v_\Omega\|^2_{\mathcal{L}^2(\Omega)}\left(\frac{\|Dv_\Omega\|^2_{\mathcal{L}^2(\Omega)}}{\|v_\Omega\|^2_{\mathcal{L}^2(\Omega)}} - \lambda_1(\Omega)\right) > 0.\end{aligned}$$

So there exists $x_0 \in \Omega$ such that

$$T(\Omega)\varphi_1^2(x_0) - \lambda_1(\Omega)v_\Omega^2(x_0) > 0.$$

Let $\Omega_{\delta,x_0} = \Omega\backslash B(x_0;\delta)$, where $B(x_0;\delta)$ is the closed ball of radius $\delta > 0$ centered at x_0. We want to show that if δ is small enough, then $F(\Omega_{\delta,x_0}) > F(\Omega)$. In the planar case $m = 2$, a classical asymptotic formula (see, for instance, [8, Theorem 1.4.1] and the references therein) gives that

$$\lambda_1(\Omega_{\delta,x_0}) = \lambda_1(\Omega) + \frac{2\pi}{-\log\delta}\varphi_1^2(x_0) + o\left(\frac{1}{|\log\delta|}\right), \quad \delta \downarrow 0. \tag{1.11}$$

Moreover, from [12, Theorem 8.1.6], we have that

$$T(\Omega_{\delta,x_0}) = T(\Omega) - \frac{2\pi}{-\log\delta}v_\Omega^2(x_0) + o\left(\frac{1}{|\log\delta|}\right), \quad \delta \downarrow 0. \tag{1.12}$$

By (1.11) and (1.12), we have that

$$\begin{aligned}\frac{T(\Omega_{\delta,x_0})\lambda_1(\Omega_{\delta,x_0})}{|\Omega_{\delta,x_0}|} &= \frac{T(\Omega)\lambda_1(\Omega)}{|\Omega|} + \frac{2\pi}{(-\log\delta)\,|\Omega|}\\ &\quad\times\left(T(\Omega)\varphi_1^2(x_0) - \lambda_1(\Omega)v_\Omega^2(x_0)\right) + o\left(\frac{1}{|\log\delta|}\right), \delta \downarrow 0.\end{aligned}$$

Hence $F(\Omega_{\delta,x_0}) > F(\Omega)$ for δ sufficiently small. The same calculation works in the higher dimensional setting ($m > 2$) replacing $\frac{2\pi}{-\log\delta}$ by the Newtonian capacity of $B(x_0;\delta)$ in (1.11) and (1.12) (see for example [8, Theorem 1.4.1] and [12, Theorem 8.1.4], respectively).

This paper is organised as follows. In Sect. 2 we prove Theorem 1.1. In Sect. 3 we prove Theorem 1.2, and in Sect. 4 we prove Theorems 1.4 and 1.5 respectively.

2. Proof of Theorem 1.1

Let v_Ω be the torsion function of Ω. By choosing v_Ω as a test function for the Rayleigh quotient for $\lambda_1(\Omega)$, we obtain that

$$\lambda_1(\Omega) \le \frac{\displaystyle\int_\Omega v_\Omega}{\displaystyle\int_\Omega v_\Omega^2}.$$

Hence

$$F(\Omega) \le \frac{\left(\displaystyle\int_\Omega v_\Omega\right)^2}{\left(\displaystyle\int_\Omega v_\Omega^2\right)|\Omega|}.$$

Let $M = \sup_\Omega v_\Omega$. For $\theta \in [0, M]$, we define

$$\mu(\theta) = |\{x \in \Omega : v_\Omega(x) > \theta\}|.$$

We have that

$$\int_\Omega v_\Omega = \int_0^M \mu(\theta)\, d\theta,$$

and

$$\int_\Omega v_\Omega^2 = \int_0^M 2\theta\mu(\theta)\, d\theta.$$

For every $\theta \in (0, M)$, we have that

$$\mu(\theta) \le (|\Omega|^{2/m} - 2m\omega_m^{2/m}\theta)^{m/2}. \tag{2.1}$$

Indeed, since v_Ω satisfies the torsion equation (1.1) in Ω, arguing similarly to [16], we have that for $\theta \in (0, M)$,

$$\mu(\theta) = \int_{\{v_\Omega=\theta\}} |Dv_\Omega|\, d\mathcal{H}^{m-1}, \tag{2.2}$$

and

$$-\mu'(\theta) \ge \int_{\{v_\Omega=\theta\}} \frac{1}{|Dv_\Omega|}\, d\mathcal{H}^{m-1}.$$

Denote the perimeter of a measurable set A by $\mathrm{Per}(A)$. Applying Hölder's inequality to $\mathrm{Per}(\{v_\Omega > \theta\}) = \int_{\{v=\theta\}} d\mathcal{H}^{m-1}$, we obtain that

$$\mathrm{Per}(\{v_\Omega > \theta\})^2 \le \mu(\theta)(-\mu'(\theta)). \tag{2.3}$$

By the isoperimetric inequality we have that

$$\mathrm{Per}(\{v_\Omega > \theta\}) \ge m\omega_m^{1/m}\mu(\theta)^{(m-1)/m}.$$

This, together with (2.3), gives the differential inequality

$$m^2\omega_m^{2/m} \le -\mu(\theta)^{\frac{2}{m}-1}\mu'(\theta).$$

Integrating this differential inequality gives (2.1).

For $t \in [0, M]$, define

$$Q(t) = \left(\int_0^t \mu(\theta)\, d\theta\right)^2 - 2\left(\int_0^t \theta\mu(\theta)\, d\theta\right)|\Omega|. \tag{2.4}$$

Using (2.1) and (2.4), it is straightforward to verify that

$$Q'(t) \leq \frac{|\Omega|^{\frac{m+2}{m}}}{m(m+2)\omega_m^{2/m}} \left[1 - \left(1 - \frac{2m\omega_m^{2/m}}{|\Omega|^{2/m}} t\right)^{\frac{m+2}{2}}\right] 2\mu(t) - 2t\mu(t)|\Omega|.$$

The inequality $(1+y)^\alpha \geq 1 + \alpha y + y^2, \alpha \geq 2,\ y \geq -1$ then gives that

$$Q'(t) \leq -|\Omega|^{1-\frac{2}{m}} \frac{8m\omega_m^{2/m}}{m+2} \mu(t)t^2. \tag{2.5}$$

Integrating (2.5) over $[0, M]$ and using the fact that $Q(0) = 0$ gives that

$$Q(M) \leq -|\Omega|^{1-\frac{2}{m}} \frac{8m\omega_m^{2/m}}{m+2} \int_0^M \mu(t)t^2\, dt.$$

Hölder's inequality then yields that

$$Q(M) \leq -|\Omega|^{1-\frac{2}{m}} \frac{2m\omega_m^{2/m}}{m+2} \frac{\left(\int_0^M 2t\mu(t)dt\right)^2}{\int_0^M \mu(t)dt} = -\frac{2m\omega_m^{2/m}}{m+2} |\Omega|^{1-\frac{2}{m}} \frac{\left(\displaystyle\int_\Omega v_\Omega^2\right)^2}{\displaystyle\int_\Omega v_\Omega}.$$

Using the expression for Q and Hölder's inequality gives that

$$\left[\frac{\left(\displaystyle\int_\Omega v_\Omega\right)^2}{\left(\displaystyle\int_\Omega v_\Omega^2\right)|\Omega|} - 1\right] \leq -\frac{2m\omega_m^{2/m}}{m+2} \frac{T(\Omega)}{|\Omega|^{1+\frac{2}{m}}}.$$

This concludes the proof of Theorem 1.1. □

3. Proof of Theorem 1.2

In this section we provide an example of an open connected set Ω_ϵ in $\mathbb{R}^m$ which satisfies (1.4). As the technical tools depend heavily on the relation between torsional rigidity and heat equation we recall some of the essential ingredients in Sect. 3.1 below. The necessary bounds for the first eigenfunction and eigenvalue with Dirichlet boundary conditions on a ball centred in an m-dimensional cube with Neumann boundary conditions will be obtained in Sect. 3.2. The proof of Theorem 1.2 will be deferred to Sect. 3.3.

3.1. Heat Equation and Torsional Rigidity

We denote the Dirichlet heat kernel for Ω by $p_\Omega(x, y; t)$, $x, y \in \Omega$, $t > 0$. The integral defined by

$$u_\Omega(x; t) = \int_\Omega dy\, p_\Omega(x, y; t)$$

is the solution of

$$\frac{\partial u(x;t)}{\partial t} = \Delta u(x;t),\, x \in \Omega,\, t > 0, \tag{3.1}$$

$$\lim_{t \downarrow 0} u(\cdot;t) = 1 \,\text{in}\, \mathcal{L}^1(\Omega), \tag{3.2}$$

$$u(\cdot;t) \in H_0^1(\Omega),\, t > 0. \tag{3.3}$$

The interpretation of (3.1), (3.2), and (3.3) is that $u_\Omega(x;t)$ represents the temperature at point x at time t when the initial temperature in Ω is 1 and the temperature of $\partial\Omega$ is 0 for all $t > 0$. The heat content of Ω at time t is defined as

$$H_\Omega(t) = \int_\Omega u_\Omega(x;t)\, dx.$$

The Dirichlet heat kernel for Ω has the following eigenfunction expansion:

$$p_\Omega(x,y;t) = \sum_{j\in\mathbb{N}} e^{-t\lambda_j(\Omega)} \varphi_j(x)\varphi_j(y). \tag{3.4}$$

It follows from Parseval's identity that

$$H_\Omega(t) = \sum_{j\in\mathbb{N}} e^{-t\lambda_j(\Omega)} \left(\int_\Omega \varphi_j\right)^2 \le e^{-t\lambda_1(\Omega)} \sum_{j\in\mathbb{N}} \left(\int_\Omega \varphi_j\right)^2 = e^{-t\lambda_1(\Omega)}|\Omega|. \tag{3.5}$$

The solution of (1.1) is given by

$$v_\Omega(x) = \int_0^\infty u_\Omega(x;t)\, dt.$$

It follows that

$$T(\Omega) = \int_0^\infty H_\Omega(t)\, dt, \tag{3.6}$$

i.e., the torsional rigidity is the integral of the heat content. By the first identity in (3.5), (3.6), and Fubini's theorem we have that

$$\begin{aligned} T(\Omega) &= \sum_{j\in\mathbb{N}} \lambda_j(\Omega)^{-1} \left(\int_\Omega \varphi_j\right)^2 \\ &\le \lambda_1(\Omega)^{-1} \sum_{j\in\mathbb{N}} \left(\int_\Omega \varphi_j\right)^2 \\ &= \lambda_1(\Omega)^{-1}|\Omega|, \end{aligned} \tag{3.7}$$

where we have used Parseval's identity in the last equality above. This implies Pólya's bound (1.3). The bound also follows by (3.5) and (3.6).

By the first identity in (3.7) we obtain that

$$T(\Omega) \ge \lambda_1(\Omega)^{-1} \left(\int_\Omega \varphi_1\right)^2.$$

3.2. Eigenfunction and Eigenvalue Bounds

We introduce the following notation. Let $\Omega_L = (-\frac{L}{2}, \frac{L}{2})^m$ be an open cube in $\mathbb{R}^m$ with measure L^m, and let K be a compact subset of Ω_L. We denote the first eigenvalue of the Laplacian acting in $\mathcal{L}^2(\Omega_L - K)$ with Neumann boundary conditions on $\partial\Omega_L$ and Dirichlet boundary conditions on ∂K by $\mu_{1,K,L}$. We denote the corresponding normalised eigenfunction by $\varphi_{1,K,L}$.

The following shows that the $\mathcal{L}^1$ norm of the first eigenfunction converges to $L^{m/2}$ as $\mu_{1,K,L} \downarrow 0$.

Lemma 3.1. *If* $m = 2, 3, 4, \ldots,$ *then*

$$L^m \left(1 - \left(\frac{4mL^2\mu_{1,K,L}}{3e}\right)^{1/2}\right) \le \|\varphi_{1,K,L}\|^2_{\mathcal{L}^1(\Omega_L - K)} \le L^m. \tag{3.8}$$

Proof. To prove (3.8), we note that by Cauchy–Schwarz,

$$\|\varphi_{1,K,L}\|^2_{\mathcal{L}^1(\Omega_L - K)} \le |\Omega_L - K| \le |\Omega_L| = L^m. \tag{3.9}$$

This proves the right-hand side of (3.8). To prove the left-hand side of (3.8), we denote the heat kernel with Neumann boundary conditions on $\partial\Omega_L$ and Dirichlet boundary conditions on ∂K by $\pi_{K,L}(x, y; t)$. By the eigenfunction expansion of $\pi_{K,L}(x, y; t)$, we have for $t > 0$ that

$$\begin{aligned} e^{-t\mu_{1,K,L}} \varphi_{1,K,L}(x)^2 &\le \pi_{K,L}(x, x; t) \le \pi_{\Omega_L}(x, x; t) \\ &\le L^{-m} \left(1 + 2\sum_{j=1}^{\infty} e^{-t\pi^2 j^2/L^2}\right)^m \\ &\le L^{-m} \left(1 + \sum_{j=1}^{\infty} \frac{2L^2}{et\pi^2 j^2}\right)^m \\ &= L^{-m} \left(1 + \frac{L^2}{3et}\right)^m, \end{aligned}$$

where $\pi_{\Omega_L}(x, y; t)$ is the Neumann heat kernel for the cube Ω_L, and where we have used the eigenfunction expansion of the latter together with separation of variables. Taking the supremum over all $x \in \Omega_L - K$ gives that

$$\|\varphi_{1,K,L}\|^2_{\mathcal{L}^\infty(\Omega_L - K)} \le e^{t\mu_{1,K,L}} L^{-m} \left(1 + \frac{L^2}{3et}\right)^m.$$

Furthermore, since $\|\varphi_{1,K,L}\|^2_{\mathcal{L}^2(\Omega_L - K)} = 1$, we have by the positivity of $\varphi_{1,K,L}$ that

$$\begin{aligned} \|\varphi_{1,K,L}\|^2_{\mathcal{L}^1(\Omega_L - K)} \ge \|\varphi_{1,K,L}\|^{-2}_{\mathcal{L}^\infty(\Omega_L - K)} &\ge L^m e^{-t\mu_{1,K,L}} \left(1 + \frac{L^2}{3et}\right)^{-m} \\ &\ge L^m \left(1 - t\mu_{1,K,L} - \frac{mL^2}{3et}\right). \end{aligned}$$

We choose $t > 0$ as to maximise the right-hand side above. This proves the left-hand side of (3.8). □

In the sequel we need upper and lower bounds for the first Dirichlet eigenvalue $\mu_{1,K,L}$ where $K = B(0;\delta) \subset \Omega_L$. These were obtained for general compact sets $K \subset \Omega_L \subset \mathbb{R}^m$, $m = 3, 4, \ldots$ in [17,18] in terms of the Newtonian capacity $\mathrm{cap}\,(K)$ of K in $\mathbb{R}^m$. The various m-dependent constants in [17, Propositions 2.2, 2.3, 2.4] and in [18, Theorem A] have not been evaluated. We supply these in the Lemmas 3.2 and 3.3 below. We consider general compact subsets as the proofs (for $m = 3, \ldots$) are hardly more involved than the special case of a ball.

Lemma 3.2. (i) *If $m = 3, 4, \ldots$ and if $K \subset \Omega_L$, then*

$$\mu_{1,K,L} \geq k_m \frac{\mathrm{cap}\,(K)}{L^m}, \tag{3.10}$$

where

$$k_m = \int_0^1 ds\,(4\pi s)^{-m/2} e^{-m/(4s)}. \tag{3.11}$$

(ii) *If $m = 3, 4, \ldots$ and if $K \subset \Omega_L$ with $\mathrm{cap}\,(K) \leq \frac{1}{16} L^{m-2}$, then*

$$\mu_{1,K,L} \leq 2\pi m \frac{\mathrm{cap}\,(K)}{L^m}. \tag{3.12}$$

Proof. By the $\mathcal{L}^2$-eigenfunction expansion of $\pi_{K,L}(x, y; t)$ we have that

$$e^{-t\mu_{1,K,L}} \varphi_{1,K,L}(x) = \int_{\Omega_L - K} dy\, \pi_{K,L}(x, y; t) \varphi_{1,K,L}(y). \tag{3.13}$$

As in [17,18], we introduce some Brownian motion tools. Let $(\tilde{B}(s), s \geq 0; \tilde{\mathbb{P}}_x, x \in \overline{\Omega_L})$ be Brownian motion with reflection on $\partial\Omega_L$. For a compact subset $K \subset \Omega_L$ we let

$$\tilde{\tau}_K = \inf\{s \geq 0 : \tilde{B}(s) \in K\}. \tag{3.14}$$

Then

$$\tilde{\mathbb{P}}_x[\tilde{\tau}_K > t] = \int_{\Omega_L - K} dy\, \pi_{K,L}(x, y; t), \tag{3.15}$$

Integrating both sides of (3.13) with respect to x over $\Omega_L - K$ gives, with (3.15), that

$$\begin{aligned} e^{-t\mu_{1,K,L}} \int_{\Omega_L - K} dx\, \varphi_{1,K,L}(x) &= \int_{\Omega_L - K} dy\, \tilde{\mathbb{P}}_y[\tilde{\tau}_K > t] \varphi_{1,K,L}(y) \\ &= \int_{\Omega_L - K} dx\, \varphi_{1,K,L}(x) - \int_{\Omega_L - K} dy\, \tilde{\mathbb{P}}_y[\tilde{\tau}_K \leq t] \varphi_{1,K,L}(y). \end{aligned}$$

It follows that

$$\begin{aligned} \mu_{1,K,L} &= -\frac{1}{t} \log\left(1 - \frac{\int_{\Omega_L - K} dy\, \tilde{\mathbb{P}}_y[\tilde{\tau}_K \leq t] \varphi_{1,K,L}(y)}{\int_{\Omega_L - K} dy\, \varphi_{1,K,L}(y)}\right) \\ &\geq \frac{1}{t} \frac{\int_{\Omega_L - K} dy\, \tilde{\mathbb{P}}_y[\tilde{\tau}_K \leq t] \varphi_{1,K,L}(y)}{\int_{\Omega_L - K} dy\, \varphi_{1,K,L}(y)} \\ &\geq \frac{1}{t} \inf_{x \in \Omega_L - K} \tilde{\mathbb{P}}_x[\tilde{\tau}_K \leq t]. \end{aligned}$$

Following [18, p.449], we define $\tilde{K}$ as the subset of $\mathbb{R}^m$ by the method of images, so that in each tiling L-cube of $\mathbb{R}^m$ we have a reflected image of K. Let $(B(s), s \geq 0; \mathbb{P}_x, x \in \mathbb{R}^m)$ be Brownian motion on $\mathbb{R}^m$, and define the first hitting time of a closed set A by

$$\tau_A = \inf\{s \geq 0 : B(s) \in A\}, \tag{3.16}$$

Then

$$\tilde{\mathbb{P}}_x[\tilde{\tau}_K \leq t] = \mathbb{P}_x[\tau_{\tilde{K}} \leq t] \geq \mathbb{P}_x[\tau_K \leq t].$$

For a compact set $K \subset \mathbb{R}^m$, we define the last exit time by

$$L_K = \sup\{s \geq 0 : B(s) \in K\},$$

where we put $L_K = +\infty$ if the supremum is over the empty set. Then $\mathbb{P}_x[\tau_K \leq t] \geq \mathbb{P}_x[L_K \leq t]$. By [13], we have that

$$\mathbb{P}_x[L_K < t] = \int \mu_K(dy) \int_0^t p(x, y; s) ds, \tag{3.17}$$

where

$$p(x, y; s) = (4\pi s)^{-m/2} e^{-|x-y|^2/(4s)}, \tag{3.18}$$

and where $\mu_K(dy)$ is the equilibrium measure of the compact K. Next we choose $t = L^2$. By the above, we have that

$$\mu_{1,K,L} \geq L^{-2} \inf_{x \in \Omega_L - K} \int \mu_K(dy) \int_0^{L^2} ds\, p(x, y; s). \tag{3.19}$$

For $y \in K$ and $x \in \Omega_L - K$, we have that $|x - y| \leq \mathrm{diam}(\Omega_L) = mL^2$. So, by (3.19), we conclude that

$$\mu_{1,K,L} \geq L^{-2} \int \mu_K(dy) \int_0^{L^2} ds\, (4\pi s)^{-m/2} e^{-mL^2/(4s)} = k_m \frac{\mathrm{cap}\,(K)}{L^m},$$

where k_m is given by (3.11). This proves part (i) of the lemma.

To prove part (ii) of the lemma, we follow the Remark on p. 451 in [18], and define the trial function

$$\psi(x) = 1 - \kappa_m^{-1} \int \mu_K(dy)\, |x - y|^{2-m}, \tag{3.20}$$

where

$$\kappa_m = \frac{4\pi^{m/2}}{\Gamma((m-2)/2)},$$

is the Newtonian capacity of the ball with radius 1 in $\mathbb{R}^m$. Then

$$|D\psi|(x) \leq \kappa_m^{-1}(m - 2) \int \mu_K(dy)\, |x - y|^{1-m}.$$

Hence

$$\|D\psi\|^2_{\mathcal{L}^2(\Omega_L - K)} \leq \kappa_m^{-2}(m - 2)^2 \int \mu_K(dy) \int \mu_K(dy') \int_{\mathbb{R}^m} dx\, |x - y|^{1-m} |x - y'|^{1-m}. \tag{3.21}$$

In order to compute the integral with respect to x over $\mathbb{R}^m$, we write

$$|x-y|^{1-m} = \frac{2\pi^{m/2}}{\Gamma((m-1)/2)} \int_0^\infty \frac{ds}{s^{1/2}} p(x,y;s). \tag{3.22}$$

By Tonelli's theorem, (3.22), and the semigroup property of the heat kernel, we have that

$$\begin{aligned}
&\int_{\mathbb{R}^m} dx\, |x-y|^{1-m}|x-y'|^{1-m} \\
&\quad = \left(\frac{2\pi^{m/2}}{\Gamma((m-1)/2)}\right)^2 \int_0^\infty \int_{\mathbb{R}^m} dx\, \frac{ds}{s^{1/2}} p(x,y;s) \int_0^\infty \frac{ds'}{s'^{1/2}} p(x,y';s') \\
&\quad = \left(\frac{2\pi^{m/2}}{\Gamma((m-1)/2)}\right)^2 \int_0^\infty \frac{ds}{s^{1/2}} \int_0^\infty \frac{ds'}{s'^{1/2}} p(y,y';s+s').
\end{aligned} \tag{3.23}$$

Changing variables $s=\sigma^2, s'=\sigma'^2$ gives that the right-hand side above equals

$$\begin{aligned}
&4\left(\frac{2\pi^{m/2}}{\Gamma((m-1)/2)}\right)^2 \int_0^\infty \int_0^\infty d\sigma\, d\sigma' p(y,y';\sigma^2+\sigma'^2) \\
&\quad = \frac{\pi^{(m+2)/2}}{\Gamma((m-1)/2)^2} \Gamma((m-2)/2)|y-y'|^{2-m}.
\end{aligned} \tag{3.24}$$

By (3.17) and (3.18), we have that for $y \in K$,

$$1 = \mathbb{P}_y[L_K < \infty] = \kappa_m^{-1} \int \mu_K(dy')|y-y'|^{2-m}. \tag{3.25}$$

Putting (3.21)-(3.25) together gives that

$$\|D\psi\|^2_{\mathcal{L}^2(\Omega_L - K)} \le \pi \left(\frac{\Gamma(m/2)}{\Gamma((m-1)/2)}\right)^2 \mathrm{cap}\,(K) \le \pi m\, \mathrm{cap}\,(K). \tag{3.26}$$

The last inequality in (3.26) follows from uniform bounds on the Γ function. See for example [1, 6.1.38]. We obtain a lower bound for $\|\psi\|_{\mathcal{L}^2(\Omega_L-K)}$ as follows. By (3.20),

$$\begin{aligned}
\|\psi\|^2_{\mathcal{L}^2(\Omega_L-K)} &\ge \int_{\Omega_L-K} dx (1 - 2\kappa_m^{-1} \int \mu_K(dy)\, |x-y|^{2-m}) \\
&= |\Omega_L - K| - 2\kappa_m^{-1} \int \mu_K(dy) \int_{\Omega_L-K} dx\, |x-y|^{2-m}.
\end{aligned}$$

By rearrangement, we have that

$$\int_{\Omega_L-K} dx\, |x-y|^{2-m} \le \int_{\Omega_L^*} dx\, |x|^{2-m} = 2^{-1} m \omega_m^{(m-2)/m} L^2,$$

where Ω_L^* is the ball centered at 0 with the same measure as Ω_L. Hence

$$\begin{aligned}
\|\psi\|^2_{\mathcal{L}^2(\Omega_L-K)} &\ge |\Omega_L| - |K| - \kappa_m^{-1} m \omega_m^{(m-2)/m} \mathrm{cap}\,(K) L^2 \\
&\ge L^m - |K| - \mathrm{cap}\,(K) L^2.
\end{aligned} \tag{3.27}$$

By the classical isoperimetric inequality for the Newtonian capacity of K,

$$|K| \le \omega_m \left(\frac{\mathrm{cap}\,(K)}{\kappa_m} \right)^{m/(m-2)} \le 7\mathrm{cap}\,(K)^{m/(m-2)}, \tag{3.28}$$

where, in the last inequality, we have used [1, 6.1.38] once more. From (3.27) and (3.28), we obtain that

$$\|\psi\|^2_{\mathcal{L}^2(\Omega_L - K)} \ge L^m - 7\mathrm{cap}\,(K)^{m/(m-2)} - \mathrm{cap}\,(K)L^2. \tag{3.29}$$

If $\mathrm{cap}\,(K) \le cL^{m-2}$ then the right-hand side of (3.29) is bounded from below by $L^m/2$ provided $7c^{m/(m-2)} + c \le \frac{1}{2}$. This clearly holds for all $c \le \frac{1}{16}$. So if $\mathrm{cap}\,(K) \le \frac{1}{16}L^{m-2}$, then $\|\psi\|^2_{\mathcal{L}^2(\Omega_L - K)} \ge L^m/2$. This, together with (3.26), completes the proof of (3.12). □

For the two-dimensional case we only consider $K = B(0;\delta) \subset \Omega_L$.

Lemma 3.3. *For $m = 2$ and $\delta < \frac{L}{6}$,*

$$\frac{1}{100L^2}\left(\log \frac{L}{2\delta} \right)^{-1} \le \mu_{1,B(0;\delta),L} \le \frac{8\pi}{(4-\pi)L^2}\left(\log \frac{L}{2\delta} \right)^{-1}. \tag{3.30}$$

Proof. We define

$$\psi(x) = \begin{cases} \dfrac{\log \frac{|x|}{\delta}}{\log \frac{L}{2\delta}}, \ \delta \le |x| \le \frac{L}{2}, \\ \\ 1, \ x \in \Omega_L \cap \{|x| > \frac{L}{2}\}. \end{cases}$$

Then

$$\|D\psi\|^2_{\mathcal{L}^2(\Omega_L - B(0;\delta))} = 2\pi \left(\log \frac{L}{2\delta} \right)^{-1},$$

and

$$\|\psi\|^2_{\mathcal{L}^2(\Omega_L - B(0;\delta))} \ge |\Omega_L \cap \{|x| > \tfrac{L}{2}\}| = \left(1 - \frac{\pi}{4}\right)L^2.$$

This proves the upper bound in (3.30).

To prove the lower bound we use the method of descent as in [18, p. 451], and observe that for $m = 2$, $\mu_{1,B(0;\delta),L}$ equals the bottom of the spectrum of the Laplacian with Neumann boundary conditions on the boundary of the cube $\Omega_L = (-\frac{L}{2}, \frac{L}{2})^3$, and Dirichlet boundary conditions on the cylinder $C_{L,\delta} = \{(x_1, x_2, x_3) : -\frac{L}{2} < x_1 < \frac{L}{2}, x_2^2 + x_3^2 < \delta^2\}$ of height L and radius δ through the centre of that cube. By the lower bound in Lemma 3.2 for $m = 3$, we obtain that

$$\mu_{1,B(0;\delta),L} \ge k_3 \frac{\mathrm{cap}\,(C_{L,\delta})}{L^3}. \tag{3.31}$$

It remains to find a lower bound for $\mathrm{cap}\,(C_{L,\delta})$. To that end, we follow similar arguments to the proof of [13, Proposition 3.4, pp. 67,68]. We consider the N balls $B_1, \dots, B_N$ with radii δ and centres $(-\frac{L}{2} + (2j-1)\delta, 0, 0)$, $j = 1, \dots N$ where $N = \lfloor \frac{L}{2\delta} \rfloor$. Recall that for any compact set $K \subset \mathbb{R}^3$,

$$\mathrm{cap}\,(K) = \sup \left\{ \left(\iint \frac{\mu(dx)\mu(dy)}{4\pi|x-y|} \right)^{-1} : \mu \in \mathbb{P}(K) \right\},$$

where $\mathbb{P}(K)$ is the collection of probability measures supported on K. By monotonicity, we have that

$$\mathrm{cap}\,(C_{L,\delta}) \geq \mathrm{cap}\,(\cup_{j=1}^{N} B_j).$$

To bound the latter, we let σ_j be the surface measure on the boundary of the jth ball, and let

$$\mu = \frac{1}{4\pi N\delta^2}\sum_{j=1}^{N}\sigma_j.$$

We wish to find an upper bound for the energy

$$\frac{1}{(4\pi N\delta^2)^2}\sum_{j=1}^{N}\sum_{k=1}^{N}\iint\frac{\sigma_j(dx)\sigma_k(dy)}{4\pi|x-y|}. \tag{3.32}$$

If $N = 1$ then the expression above equals the inverse of $\mathrm{cap}\,(B(0;\delta))$. The contribution from the N terms with $j = k$ in (3.32) equals $\frac{1}{4\pi N\delta}$. Furthermore, the contribution of the terms with $|j-k| = 1$ in (3.32) is bounded by $\frac{N-1}{(4\pi N\delta^2)^2}\iint\frac{\sigma_1(dx)\sigma_2(dy)}{4\pi|x-y|}$. As $\delta^{-1}d\sigma_j$ is the equilibrium measure for the jth ball, we have that $\int\frac{\delta^{-1}\sigma_2(dy)}{4\pi|x-y|} \leq 1$. We conclude that

$$\frac{N-1}{(4\pi N\delta^2)^2}\iint\frac{\sigma_1(dx)\sigma_2(dy)}{4\pi|x-y|} \leq \frac{N-1}{4\pi N^2\delta^3}\int\sigma_1(dx) \leq \frac{1}{4\pi N\delta}.$$

Similarly the contribution of the terms with $|j-k| = 2$ in (3.32) is bounded by

$$\frac{N-2}{(4\pi N\delta^2)^2}\iint\frac{\sigma_1(dx)\sigma_3(dy)}{4\pi|x-y|} \leq \frac{1}{4\pi N\delta}.$$

It remains to find an upper bound for the terms in (3.32) for $|j-k| \geq 3$. For x, y on the surface of the j, kth balls we have that $|x-y| \geq 2|k-j-1|\delta$. Hence the contribution from the terms with $|j-k| \geq 3$ in (3.32) is bounded from above by

$$\begin{aligned}
&\frac{1}{(4\pi N\delta^2)^2}\sum_{k=1}^{N}\left(\sum_{j\geq k+3}^{N}\frac{1}{8\pi(j-k-1)\delta} + \sum_{j\leq k-3}\frac{1}{8\pi(k-1-j)\delta}\right)\sigma_1(B_1)\sigma_2(B_2)\\
&\quad= \frac{1}{8\pi N^2\delta}\sum_{k=1}^{N}\left(\sum_{j\geq k+3}^{N}\frac{1}{j-k-1} + \sum_{j\leq k-3}\frac{1}{k-j-1}\right)\\
&\quad\leq \frac{1}{4\pi N^2\delta}\sum_{k=1}^{N}\sum_{j=2}^{N}\frac{1}{j}\\
&\quad\leq \frac{\log N}{4\pi N\delta}.
\end{aligned}$$

Collecting all terms, we see that the expression under (3.32) is bounded from above by $\frac{3+\log N}{4\pi N\delta}$. Hence

$$\mathrm{cap}\,(C_{L,\delta}) \geq \frac{4\pi N\delta}{3+\log N} \geq \frac{4\pi L}{3(3+\log N)} \geq L\left(\log\frac{L}{2\delta}\right)^{-1}, \tag{3.33}$$

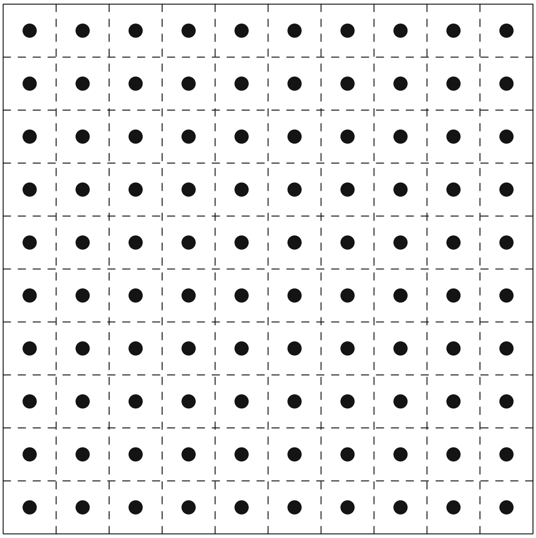

Figure 1. $\Omega_{\delta,N,L}$ with $m=2, N=10, \delta=\frac{L}{8N}$.

where we have used that $N \ge 3, \delta \le L/6$. Numerical evaluation gives that $k_3 \ge 0.0101\ldots \ge \frac{1}{100}$. The lower bound in Lemma 3.3 follows by (3.31) and (3.33). $\square$

3.3. Proof of Theorem 1.2

We partition Ω_L into N^m disjoint open cubes $C_1,\ldots,C_{N^m}$ each with measure $(L/N)^m$. We denote the centres of these cubes by $c_1,\ldots,c_{N^m}$ respectively. Let $0<\delta<\frac{L}{2N}$, and put

$$\Omega_{\delta,N,L}=\Omega_L-\cup_{i=1}^{N^m}B(c_i;\delta). \tag{3.34}$$

We denote the Dirichlet heat kernel for $\Omega_{\delta,N,L}$ and Ω_L by $p_{\Omega_{\delta,N,L}}(x,y;t)$ and $p_{\Omega_L}(x,y;t)$ respectively. The heat kernel with Neumann boundary conditions on $\partial\Omega_L$ and Dirichlet boundary conditions on $\partial\Omega_{\delta,N,L}-\partial\Omega_L$ will be denoted by $\pi_{\Omega_{\delta,N,L}}(x,y;t)$. Let $T>0$, and let $\epsilon>0$ be arbitrary. We bound the torsional rigidity for $\Omega_{\delta,N,L}$ from below as follows.

$$\begin{aligned}
T(\Omega_{\delta,N,L}) &= \int_{\Omega_{\delta,N,L}} dx \int_{\Omega_{\delta,N,L}} dy \int_0^\infty dt\, p_{\Omega_{\delta,N,L}}(x,y;t)\\
&\ge \int_{\Omega_{\delta,N,L}} dx \int_{\Omega_{\delta,N,L}} dy \int_0^T dt\, p_{\Omega_{\delta,N,L}}(x,y;t)\\
&= \int_{\Omega_{\delta,N,L}} dx \int_{\Omega_{\delta,N,L}} dy \int_0^T dt\, \pi_{\Omega_{\delta,N,L}}(x,y;t)\\
&\quad - \int_{\Omega_{\delta,N,L}} dx \int_{\Omega_{\delta,N,L}} dy \int_0^T dt\, (\pi_{\Omega_{\delta,N,L}}(x,y;t)-p_{\Omega_{\delta,N,L}}(x,y;t)).
\end{aligned} \tag{3.35}$$

We now use (3.14), (3.15) and (3.16) with $K = \cup_{i=1}^{N^m} B(c_i;\delta)$, and $A = \Omega_{\delta,N,L}$, $A = \partial\Omega_{\delta,N,L} - \partial\Omega_L$ respectively. So

$$\mathbb{P}_x[\tau_{\partial\Omega_{\delta,N,L}} > t] = \int_{\Omega_{\delta,N,L}} dy\, p_{\Omega_{\delta,N,L}}(x,y;t),$$

$$\tilde{\mathbb{P}}_x[\tilde{\tau}_{\partial\Omega_{\delta,N,L}-\partial\Omega_L} > t] = \int_{\Omega_{\delta,N,L}} dy\, \pi_{\Omega_{\delta,N,L}}(x,y;t),$$

and

$$\begin{aligned}\tilde{\mathbb{P}}_x[\tilde{\tau}_{\partial\Omega_{\delta,N,L}-\partial\Omega_L} > t] &= \tilde{\mathbb{P}}_x[\tilde{\tau}_{\partial\Omega_{\delta,N,L}} > t] + \tilde{\mathbb{P}}_x[\tilde{\tau}_{\partial\Omega_L} < t < \tilde{\tau}_{\partial\Omega_{\delta,N,L}}]\\ &\le \mathbb{P}_x[\tau_{\partial\Omega_{\delta,N,L}} > t] + \tilde{\mathbb{P}}_x[\tilde{\tau}_{\partial\Omega_L} < t]\\ &= \mathbb{P}_x[\tau_{\partial\Omega_{\delta,N,L}} > t] + \mathbb{P}_x[\tau_{\partial\Omega_L} < t].\end{aligned}$$

Hence the second term in the right-hand side of (3.35) is bounded in absolute value by

$$\begin{aligned}\int_{\Omega_{\delta,N,L}} dx \int_0^T dt\, \mathbb{P}_x[\tau_{\partial\Omega_L} < t] &\le \int_{\Omega_L} dx \int_0^T dt\, \mathbb{P}_x[\tau_{\partial\Omega_L} < t]\\ &\le \int_{\Omega_L} dx \int_0^T dt\, \mathbb{P}_x[\tau_{\partial B(x;d(x))} < t]\\ &\le 2^{(m+2)/2} \int_0^T dt \int_{\Omega_L} dx\, e^{-d(x)^2/(8t)}\\ &\le 2^{(m+2)/2}\mathcal{H}^{m-1}(\partial\Omega_L) \int_0^T dt \int_0^\infty dr\, e^{-r^2/(8t)}\\ &= s_m L^{m-1} T^{3/2}, \end{aligned} \tag{3.36}$$

with

$$s_m = 2^{(m+7)/2} m\pi^{1/2}/3.$$

In (3.36) we denoted by $d(x) = \min\{|x-y| : y \in \partial\Omega_L\}$, and by $\mathcal{H}^{m-1}(\partial\Omega_L)$ the surface area of $\partial\Omega_L$. In the third inequality of (3.36) we used Corollary [5, 6.4], while the fourth inequality follows from the fact that parallel sets of a convex set have decreasing surface area. See [6, Proposition 2.4.3].

By the periodicity of the cooling obstacles in $\Omega_{\delta,N,L}$ and the fact that we have no heat flow across $\partial\Omega_L$ we conclude that

$$\begin{aligned}&\int_{\Omega_{\delta,N,L}} dx \int_{\Omega_{\delta,N,L}} dy\ \pi_{\Omega_{\delta,N,L}}(x,y;t)\\ &\qquad = N^m \int_{C_1 - B(c_1;\delta)} dx \int_{C_1 - B(c_1;\delta)} dy\, \pi_{B(c_1;\delta),C_1}(x,y;t),\end{aligned}$$

where $\pi_{B(c_1;\delta),C_1}(x,y;t)$ denotes the heat kernel with Neumann boundary conditions on ∂C_1 and Dirichlet boundary conditions on $\partial B(c_1;\delta)$. We denote the spectral resolution of the corresponding Laplace operator acting in $\mathcal{L}^2(C_1 - B(c_1;\delta))$ by $\{\mu_{j,B(c_1;\delta),L/N}, j = 1,2,\ldots\}$, and denote a corresponding orthonormal basis of eigenfunctions by $\{\varphi_{j,B(c_1;\delta),L/N}, j = 1,2,\ldots\}$. Using the spectral resolution as in (3.4) we have that

$$
\begin{aligned}
&\int_{C_1-B(c_1;\delta)} dx \int_{C_1-B(c_1;\delta)} dy\, \pi_{B(c_1;\delta),C_1}(x,y;t) \\
&\qquad\qquad = \sum_{j\in\mathbb{N}} e^{-t\mu_{j,B(c_1;\delta),L/N}} \left(\int_{C_1-B(c_1;\delta)} \varphi_{j,B(c_1;\delta),L/N}\right)^2 \\
&\qquad\qquad \ge e^{-t\mu_{1,B(c_1;\delta),L/N}} \|\varphi_{1,B(c_1;\delta),L/N}\|^2_{\mathcal{L}^1(C_1-B(c_1;\delta))}.
\end{aligned}
$$

We conclude that the first term in the left-hand side of (3.35) is bounded from below by

$$
\begin{aligned}
&N^m \mu^{-1}_{1,B(c_1;\delta),L/N}(1-e^{-T\mu_{1,B(c_1;\delta),L/N}})\|\varphi_{1,B(c_1;\delta),L/N}\|^2_{\mathcal{L}^1(C_1-B(c_1;\delta))} \\
&\quad \ge N^m \mu^{-1}_{1,B(c_1;\delta),L/N}\|\varphi_{1,B(c_1;\delta),L/N}\|^2_{\mathcal{L}^1(C_1-B(c_1;\delta))} \\
&\quad\quad -\mu^{-1}_{1,B(c_1;\delta),L/N} L^m e^{-T\mu_{1,B(c_1;\delta),L/N}},
\end{aligned}
\tag{3.37}
$$

where we have used (3.9). By (3.35), (3.36) and (3.37) we have that

$$
\begin{aligned}
T(\Omega_{\delta,N,L}) \ge{}& N^m \mu^{-1}_{1,B(c_1;\delta),L/N}\|\varphi_{1,B(c_1;\delta),L/N}\|^2_{\mathcal{L}^1(C_1-B(c_1;\delta))} \\
&-\mu^{-1}_{1,B(c_1;\delta),L/N} L^m e^{-T\mu_{1,B(c_1;\delta),L/N}} - s_m L^{m-1} T^{3/2}.
\end{aligned}
\tag{3.38}
$$

By Dirichlet–Neumann bracketing ([15]), we first replace the Dirichlet boundary conditions on $\partial\Omega_L$ by Neumann boundary conditions and we subsequently insert Neumann boundary conditions on all of the boundaries of the cubes C_i. This gives that $\lambda_1(\Omega_{\delta,N,L}) \ge \mu_{1,B(c_1;\delta),L/N}$. Furthermore $|\Omega_{\delta,N,L}| \le L^m$. So by (1.2), (3.38) and Lemma 3.1, we obtain that

$$
\begin{aligned}
F(\Omega_{\delta,N,L}) \ge{}& \frac{N^m}{L^m}\|\varphi_{1,B(c_1;\delta),L/N}\|^2_{\mathcal{L}^1(C_1-B(c_1;\delta))} \\
&-e^{-T\mu_{1,B(c_1;\delta),L/N}} - s_m \mu_{1,B(c_1;\delta),L/N} L^{-1} T^{3/2} \\
\ge{}& 1 - \left(\frac{4mL^2\mu_{1,B(c_1;\delta),L/N}}{3N^2 e}\right)^{1/2} \\
&-e^{-T\mu_{1,B(c_1;\delta),L/N}} - s_m \mu_{1,B(c_1;\delta),L/N} L^{-1} T^{3/2}.
\end{aligned}
\tag{3.39}
$$

We now choose

$$
T = \frac{\log N}{\mu_{1,B(c_1;\delta),L/N}}. \tag{3.40}
$$

This gives by (3.39), (3.40) that

$$
F(\Omega_{\delta,N,L}) \ge 1 - \left(\frac{4mL^2\mu_{1,B(c_1;\delta),L/N}}{3eN^2}\right)^{1/2} - N^{-1} - s_m \frac{(\log N)^{3/2}}{L\mu^{1/2}_{1,B(c_1;\delta),L/N}}. \tag{3.41}
$$

We first consider the case $m \ge 3$, and use the bound in (3.12) to obtain that for

$$
\kappa_m \left(\frac{N\delta}{L}\right)^{m-2} \le \frac{1}{16}, \tag{3.42}
$$

$$
\left(\frac{4mL^2\mu_{1,B(c_1;\delta),L/N}}{3eN^2}\right)^{1/2} \le \left(\frac{8\pi m^2\kappa_m}{3e}\right)^{1/2}\left(\frac{N\delta}{L}\right)^{(m-2)/2}. \tag{3.43}
$$

Similarly (3.10) gives that

$$\frac{L}{N}\mu_{1,B(c_1;\delta),L/N}^{1/2} \geq (k_m\kappa_m)^{1/2}\left(\frac{N\delta}{L}\right)^{(m-2)/2}. \tag{3.44}$$

Combining (3.41), (3.43) and (3.44), we see that the right-hand side of (3.41) is of the form

$$F(\Omega_{\delta,N,L}) \geq 1 - N^{-1} - \left(\frac{8\pi m^2\kappa_m}{3e}\right)^{1/2}\theta - \frac{s_m}{(k_m\kappa_m)^{1/2}}\frac{(\log N)^{3/2}}{N\theta}. \tag{3.45}$$

with

$$\theta = \left(\frac{N\delta}{L}\right)^{(m-2)/2}.$$

The choice $\theta = \frac{(\log N)^{3/4}}{N^{1/2}}$ gives that

$$F(\Omega_{\delta,N,L}) \geq 1 - O\left(\frac{(\log N)^{3/4}}{N^{1/2}}\right),\ N \to \infty. \tag{3.46}$$

For $m = 2$ we obtain by (3.41) and Lemma 3.3 that

$$F(\Omega_{\delta,N,L}) \geq 1 - N^{-1} - \left(\frac{32}{3e(4-\pi)}\right)^{1/2}\theta - \frac{300 s_2(\log N)^{3/2}}{\pi N}\theta^{-1}, \tag{3.47}$$

with

$$\theta = \left(\log\frac{L}{2N\delta}\right)^{-1/2}.$$

Maximising the right-hand side of (3.47) with respect to θ yields again (3.46) after a straightforward calculation. The assertion in Theorem 1.2 follows by taking $\Omega_\epsilon = C_{\delta,N,L}$ where δ, N and L satisfy the above relations (for $m = 2$ and $m \geq 3$) for the optimal choice of θ, and by choosing $N \in \mathbb{N}$ so large that the term $(\log N)^{3/4}/N^{1/2}$ in (3.46) is smaller than ϵ. □

4. Proofs of Theorem 1.4 and 1.5

In this section we give the proofs of Theorem 1.4 and Theorem 1.5 respectively.

Proof of Theorem 1.4. By the John's ellipsoid Theorem ([9]), there exists an ellipsoid Υ with centre c such that $\Upsilon \subset \Omega \subset c+m(\Upsilon - c)$. Here $c+m(\Upsilon-c) = \{c+m(x-c) : x \in \Upsilon\}$. This is the dilation of Υ by a factor of m with centre c. Υ is the ellipsoid of maximal volume in Ω. By translating both Ω and Υ we may assume that

$$\Upsilon = \left\{x \in \mathbb{R}^m : \sum_{i=1}^m \frac{x_i^2}{a_i^2} < 1\right\}, \quad a_i > 0, \quad i = 1, \dots, m.$$

It is easily verified that the unique solution of (1.1) for Υ is given by

$$v_\Upsilon(x) = 2^{-1}\left(\sum_{i=1}^m \frac{1}{a_i^2}\right)^{-1}\left(1 - \sum_{i=1}^m \frac{x_i^2}{a_i^2}\right).$$

By changing to spherical coordinates, we find that

$$T(\Omega) \geq T(\Upsilon) = \int_{\Upsilon} v_{\Upsilon} = \frac{\omega_m}{m+2} \left(\sum_{i=1}^{m} \frac{1}{a_i^2} \right)^{-1} \prod_{i=1}^{m} a_i. \tag{4.1}$$

Since $\Omega \subset m\Upsilon$,

$$|\Omega| \leq \int_{m\Upsilon} dx = \omega_m m^m \prod_{i=1}^{m} a_i. \tag{4.2}$$

By monotonicity of Dirichlet eigenvalues, we have that $\lambda_1(\Omega) \geq \lambda_1(m\Upsilon)$. The ellipsoid $m\Upsilon$ is contained in a cuboid with lengths $2ma_1, \dots, 2ma_m$. So we have that

$$\lambda_1(\Omega) \geq \frac{\pi^2}{4m^2} \sum_{i=1}^{m} \frac{1}{a_i^2}. \tag{4.3}$$

Combining (4.1), (4.2) and (4.3) gives the lower bound in (1.6).

To prove part (ii) we note (see [10]) that for bounded, convex Ω in $\mathbb{R}^2$,

$$\lambda_1(\Omega) \geq \frac{\pi^2 \mathrm{Per}(\Omega)^2}{16|\Omega|^2}. \tag{4.4}$$

Furthermore, by [7, Theorem 5.1], we have that for Ω convex in $\mathbb{R}^m$,

$$T(\Omega) \geq \frac{|\Omega|^3}{3\mathrm{Per}(\Omega)^2}. \tag{4.5}$$

The assertion under (1.7) follows by (4.4) and (4.5). □

Proof of Theorem 1.5. We claim that it is always possible to choose $z_1, z_2 \in \partial\Omega$ such that $|z_1 - z_2| = w$, and therefore the vector $z_1 - z_2$ is orthogonal at z_1 and z_2 to two parallel supporting hyperplanes achieving the minimal distance w.

To show this, the first step is to prove that for any direction ν, there exist two points $\tilde{z}_1, \tilde{z}_2 \in \partial\Omega$ such that the supporting hyperplanes tangent to $\partial\Omega$ at these points are parallel to each other. Indeed, assuming that the set is smooth and strictly convex (the general case follows at once from an approximation argument), for every $\eta \in \mathcal{S}^{m-1}$ such that $\eta \cdot \nu > 0$, there exists a unique point $\tilde{x}(\eta) \in \partial\Omega$ where the outer unit normal is η. Moreover, there exists a unique point $\bar{x}(\tilde{x}) \in \partial\Omega$ such that $\tilde{x} - \bar{x}$ is parallel to ν. We denote by $\xi(\tilde{x})$ the inner unit normal to Ω at $\tilde{x}$ and observe that $\xi \cdot \nu > 0$. Therefore, denoting by $\mathcal{S}_\nu = \{\eta \in \mathcal{S}, \eta \cdot \nu \geq 0\}$, the map $\xi(\tilde{x}(\bar{x}(\eta)))$ (possibly extended so that $\xi = -\eta$ when $\eta \cdot \nu = 0$) is a continuous map from $\mathcal{S}_\nu$ into itself. Brouwer's fixed point theorem provides the existence of $\hat{\eta}$ such that $\xi(\hat{\eta}) = \hat{\eta}$ and this completes the first step. Now, in view of the above result, assuming that T_1 and T_2 are two supporting hyperplanes at distance w, there exist two points $z_1, z_2 \in \partial\Omega$ such that $z_1 - z_2$ is orthogonal to T_1 and T_2, and the supporting hyperplanes tangent to $\partial\Omega$ at z_1 and z_2 are parallel to each other. On one hand we have $w \leq |z_1 - z_2|$, and on the other hand, by construction, $|z_1 - z_2|$, is not greater than the distance between T_1 and T_2. This forces z_1 and z_2 to belong to T_1 and T_2 and hence $w = |z_1 - z_2|$, which proves our claim.

We introduce a reference frame in $\mathbb{R}^m$, $(x,y) \in \mathbb{R}\times\mathbb{R}^{m-1}$ where x points in the direction $z_1 - z_2$ and $(0,0) = \frac{z_1+z_2}{2}$. Denoting by E the projection of Ω onto the hyperplane $x=0$, we have

$$\Omega = \{(x,y) \in \mathbb{R}^m : l(y) < x < L(y),\, y \in E\}, \tag{4.6}$$

where $L : E \mapsto \mathbb{R}$ is concave, $l : E \mapsto \mathbb{R}$ is convex, $l \le L$ and $\max\{L(y)-l(y) : y \in E\} = w$. This maximum is achieved at $y=0$.

We note that $\{(x,y) \in \mathbb{R}^m : x = 0, (x,y) \in \Omega\} \supset \frac{1}{2}E$, where $\frac{1}{2}E$ is the homothety of E by $\frac{1}{2}$ with respect to $y=0$. We consider the two-sided cone with base $\frac{1}{2}E$ and vertices $(\frac{w}{2},0)$ and $(\frac{-w}{2},0)$. Let $h \in [0,\frac{w}{2}]$. This two-sided cone contains a cylinder C_h with height $2h$ and base $\left(1-\frac{2h}{w}\right)\frac{1}{2}E$. By monotonicity of Dirichlet eigenvalues, we have that $\lambda_1(\Omega) \le \lambda_1(C_h)$. By separation of variables, we have that

$$\lambda_1(\Omega) \le \frac{\pi^2}{4h^2} + \frac{4w^2\Lambda}{(w-2h)^2}. \tag{4.7}$$

Minimising the right-hand side of (4.7) with respect to h gives that

$$\lambda_1(\Omega) \le \frac{\pi^2}{w^2}\left(1 + \frac{3c}{2} + \frac{3c^2}{4} + \frac{c^3}{8}\right), \tag{4.8}$$

where c is given by (1.9).

If we denote the torsion function of Ω by $v = v(x,y)$ where $x \in \mathbb{R}$ and $y \in \mathbb{R}^{m-1}$, then

$$\begin{aligned}\frac{|\Omega|}{T(\Omega)} &= |\Omega|\frac{\displaystyle\int_E\left(\int_l^L (|D_y v|^2 + v_x^2)\,dx\right)dy}{\left(\displaystyle\int_E\left(\int_l^L v\,dx\right)dy\right)^2} \ge |\Omega|\frac{\displaystyle\int_E\left(\int_l^L v_x^2\,dx\right)dy}{\left(\displaystyle\int_E\left(\int_l^L v\,dx\right)dy\right)^2}\\ &\ge \inf_{\phi}|\Omega|\frac{\displaystyle\int_E\left(\int_l^L \phi_x^2\,dx\right)dy}{\left(\displaystyle\int_E\left(\int_l^L \phi\,dx\right)dy\right)^2} = \frac{\displaystyle\int_E (L-l)\,dy}{\displaystyle\int_E \frac{(L-l)^3}{12}\,dy}.\end{aligned} \tag{4.9}$$

The last equality follows by the fact that the function

$$(x,y) \mapsto \frac{1}{2}\left\{\left(\frac{L(y)-l(y)}{2}\right)^2 - \left(x - \frac{L(y)+l(y)}{2}\right)^2\right\}$$

achieves the minimum. We conclude that

$$\frac{T(\Omega)}{|\Omega|} \le \frac{1}{12}\frac{\displaystyle\int_E (L-l)^3\,dy}{\displaystyle\int_E (L-l)\,dy} \le \frac{w^2}{12}. \tag{4.10}$$

Combining (4.8) with (4.10) gives (1.8).

To prove part (ii), we note that for $m = 2$ Theorem 1.1 gives that for any Ω with finite Lebesgue measure,

$$F(\Omega) \le 1 - \frac{\pi}{\lambda_1(\Omega)|\Omega| + \pi}.$$

By Blaschke's theorem, [20, p. 215], Ω contains a ball with radius $w/3$. Hence $\lambda_1(\Omega) \le 9{j_{0,1}}^2/w^2$, where $j_{0,1} = 2.405\ldots$ is the first positive zero of the Bessel function J_0. Furthermore, since $|\Omega| \le w|E|$ and $|E| \ge w$, we have that

$$F(\Omega) \le 1 - \frac{\pi}{\pi + 9j_{0,1}^2}\left(\frac{w}{|E|}\right). \tag{4.11}$$

For $\frac{w}{|E|}$ small we use part (i) to obtain an upper bound. Since

$$\Lambda = \frac{\pi^2}{|E|^2} \le \frac{\pi^2}{w^2}, \tag{4.12}$$

we have, by (1.9), that $c \le (32)^{1/3}$. By (1.8) and (4.12), we get that

$$\begin{aligned} F(\Omega) &\le \frac{\pi^2}{12}\left(1 + \left(\frac{3}{2} + \frac{3}{2^{1/3}} + 2^{1/3}\right)c\right) \\ &= \frac{\pi^2}{12} + \left(2^{-4/3} + 2^{-2/3} + 3^{-1}\right)\pi^2\left(\frac{w}{|E|}\right)^{2/3}. \end{aligned} \tag{4.13}$$

For $\frac{w}{|E|}$ small we use (4.13) as an upper bound, while for $\frac{w}{|E|}$ large we use (4.11) as an upper bound. The cross-over point value of $\frac{w}{|E|}$ where the right-hand side of (4.11) equals the right-hand side of (4.13) is bounded from below by 0.0015197. This, together with the bound under (4.11), gives the assertion under (1.10). $\square$

Below we list some known numerical values of F for some convex planar shapes.

Shape	F(Shape)
Rectangle with sides a, b	$\frac{\pi^2}{12}(1 + O(a/b))$, $a/b \downarrow 0$, $(\frac{\pi^2}{12} \approx 0.822)$
Disc	$\frac{j_{0,1}^2}{8} \approx 0.723$
Half-disc	$\left(\frac{1}{4} - \frac{2}{\pi^2}\right)j_{1,1}^2 \approx 0.695$
Equilateral triangle	$\frac{\pi^2}{15} \approx 0.658$

In the table above $j_{1,1}$ is the first positive zero of the Bessel function J_1. The values for the thin rectangle and the disc are taken from [4]. The torsional rigidity of an equilateral triangle with side lengths a equals $\frac{\sqrt{3}a^4}{320}$ [19, pp. 263–265]. To obtain the third line in the table we note that for a half-disc of area $\pi a^2/2$, the torsional rigidity is given by $\left(\frac{\pi}{8} - \frac{1}{\pi}\right)a^4$ (see [19, pp. 265–267]). The first Dirichlet eigenvalue of the half-disc is the second Dirichlet eigenvalue of the full disc and equals $j_{1,1}^2 a^2$. The first Dirichlet eigenvalue of an equilateral triangle with side lengths a is given by $\frac{16\pi^2}{3a^2}$. So we obtain the last line in the table above.

References

[1] Abramowitz, M., Stegun, I.A.: Pocketbook of Mathematical Functions. Verlag Harri Deutsch, Thun (1984)

[2] Bandle, C.: Isoperimetric Inequalities and Applications, Monographs and Studies in Mathematics. Pitman, London (1980)

[3] van den Berg, M.: Large time asymptotics of the heat flow. Quart. J. Math. Oxf. Ser. **41**, 245–253 (1990)

[4] van den Berg, M., Buttazzo, G., Velichkov, B.: Optimization problems involving the first Dirichlet eigenvalue and the torsional rigidity. In: Pratelli, A., Leugering, G. (eds.) New Trends in Shape Optimization, Int. Series Numerical Math, vol. 166, pp. 19–41. Birkhäuser, Basel (2016)

[5] van den Berg, M., Davies, E.B.: Heat flow out of regions in $\mathbb{R}^m$. Math. Z. **202**, 463–482 (1989)

[6] Bucur, D., Buttazzo, G.: Variational Methods in Shape Optimization Problems, Progress in Nonlinear Differential Equations and their Applications 65. Birkhäuser Boston Inc, Boston (2005)

[7] Della Pietra, F., Gavitone, N.: Sharp bounds for the first eigenvalue and the torsional rigidity related to some anisotropic operators. Math. Nachr. **287**, 194–209 (2014)

[8] Henrot, A.: Extremum problems for eigenvalues of elliptic operators. Frontiers in Mathematics. Birkhäuser Verlag, Basel (2006)

[9] John, F.: Extremum problems with inequalities as subsidiary conditions. Studies and Essays Presented to R. Courant on his 60'th Birthday, January 8, 1948, pp. 187–204. Interscience Publishers, Inc., New York, NY (1948)

[10] Makai, E.: On the principal frequency of a membrane and the torsional rigidity of a beam. Studies in Mathematical Analysis and Realted Topics: Essays in Honor of George Pólya, pp. 227–231. Stanford University Press, Stanford (1962)

[11] Markvorsen, S., Palmer, V.: Torsional rigidity of minimal submanifolds. Proc. Lond. Math. Soc. **93**, 253–272 (2006)

[12] Maz'ya, V., Nazarov, S.A., Plamenevskiĭ, B.A.: Asymptotic Theory of Elliptic Boundary Value Problems in Singularly Perturbed Domains. Birkhäuser Verlag, Basel (2000)

[13] Port, S.C., Stone, C.J.: Brownian Motion and Classical Potential Theory. Academic Press, New York (1967)

[14] Pólya, G., Szegö, G.: Isoperimetric Inequalities in Mathematical Physics, Ann. of Math. Stud. 27. Princeton University Press, Princeton (1951)

[15] Reed, M., Simon, B.: Methods of Modern Mathematical Physics IV, Analysis of Operators. Academic Press, New York (1978)

[16] Talenti, G.: Elliptic equations and rearrangements. Ann. Scuola Norm. Sup. Pisa **3**, 697–718 (1976)

[17] Taylor, M.E.: Scattering length and perturbations of $-\Delta$ by positive potentials. J. Math. Anal. Appl. **53**, 291–312 (1976)

[18] Taylor, M.E.: Estimate on the fundamental frequency of a drum. Duke Math. J. **46**, 447–453 (1979)

[19] Timoshenko, S.P., Goodier, J.N.: Theory of Elasticity. McGraw-Hill Book Company Inc, New York (1951)

[20] Yaglom, I.M., Boltyanskii, V.G.: Convex figures, (Translated by P. J. Kelly, L. F. Walton), Holt, Rinehart and Winston, New York (1961)

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